7

Disparate scale interactions

Long and short delimit each other.

(Lao Tzu, "Tao Te Ching")

7.1 Short overview

In this chapter, we describe one generic class of nonlinear interaction in plasmas, called disparate scale interaction. One of the characteristic features of plasma turbulence is that there can be several explicit distinct scale lengths in the dynamics. For example, the ion gyroradius and electron gyroradius define intrinsic lengths in magnetized plasmas. The Debye length $\lambda_{\mathrm{De}} = v_{\mathrm{T},e}/\omega_{\mathrm{pe}}$ gives the boundary for collective oscillation, and the collisionless skin depth, c/ω_{pe}, is the scale of magnetic perturbation screening. These characteristic scale lengths define related modes in plasmas.

One evident reason why these scales are disparate is that the electron mass and ion mass differ substantially. Thus the fluctuations at different scale lengths can have different properties (in the dispersion, eigenvectors, etc.). Such a separation is not limited to *linear* dispersion, but occurs even in *nonlinear* dynamics. This is because the unstable modes are coupled with stable modes within the same group of fluctuations (those with a common scale length). The plasma response often leads to the result that instability is possible for a particular class of wave numbers. For instance, the drift waves in magnetized plasma (which preferentially propagate in the direction of diamagnetic drift velocity) can be unstable only if the wave number in the direction of the magnetic field $k_{\parallel}$ is much smaller than that in the direction of propagation. Therefore, nonlinear interaction within a *like-scale*, which increases $k_{\parallel}$ significantly, allows a transfer of the fluctuation energy to strongly damped modes.

This is in contrast with the familiar case of the Kolmogorov cascade in neutral fluids. In this consideration, the kinetic energy, which is contained in an observable

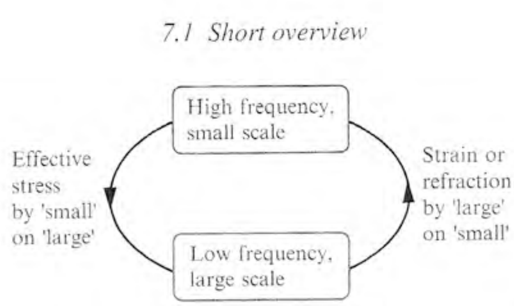

Fig. 7.1. Interaction of small-scale fluctuations and large-scale fluctuations.

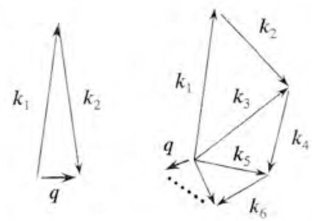

Fig. 7.2. Comparison of disparate scale interaction (left) and inverse cascade (right). The long-wavelength perturbation (characterized by $\boldsymbol{q}$) is directly generated by, e.g., parametric instability in the disparate-scale interactions.

scale L, is transferred to, and dissipated (by molecular viscosity) at, the microscale l_{d}. The ratio L/l_{d} is evaluated as $L/l_{\mathrm{d}} \simeq R_e^{3/4}$, where R_e is the Reynolds number. Between these two scales, L and l_{d}, there is no preferred scale.

The presence of multiple-scale dynamics in plasmas allows a new class of multi-scale nonlinear interaction, i.e., *disparate scale interaction*. Consider the situation that two kinds of fluctuations (with different scales) coexist, as is illustrated in Figure 7.1. The fluctuations with high frequency and small spatial scale (referred to as 'small scale') can provide a 'source' for the fluctuations with low frequency and large spatial scale (referred to as 'large scale'), by inducing a stress which acts upon them, as fluxes in their density, momentum and energy. (This mechanism is explained in Chapter 3.) On the other hand, the large-scale perturbation acts as a 'strain field' in the presence of which the small-scale fluctuation evolves.

Disparate scale interaction is a prototypical process for the structure in turbulent plasma (Diamond *et al.*, 2005b). Large-scale structure (flows, density modulation, etc.) are generated as a result of the evolution of perturbations which break the symmetry of the turbulence. This mechanism is explained in detail in this chapter. The process of 'formation of large scale by turbulence' has some similarity to the 'inverse cascade' in fluid dynamics. Of course both of them share common physics, but at least noticeable difference between them exists. In the process of the inverse-cascade, the energy transfer from the small scales to the large scales occurs by interaction between two excitations of comparable scale (Figure 7.2).

In contrast, in the process depicted in Figure 7.1, the transfer of energy does not occur through a sequence of intermediate scales, but rather proceeds directly between small and large. Thus, the concept of the *disparate-scale interaction* plays a key role in understanding structure formation.

This type of interaction has many examples. The first and simplest is the interaction of the Langmuir wave (plasma wave) and the acoustic wave (ion sound wave) (Zakharov, 1984; Goldman, 1984; Zakharov, 1985; Robinson, 1997). This is the *classic example* of disparate scale interaction, and has significantly impacted our understanding of plasma turbulence. In this case, the ponderomotive pressure associated with the amplitude modulation of Langmuir waves (at large scale) induces the depletion of plasma density. On the other hand, the density modulation associated with the acoustic wave causes refraction of plasma waves, so that the plasma waves tend to accumulate in an area of lower density. These two effects close the interaction loop (in Figure 7.1), so that the amplitude modulation of plasma waves grows in time.

The second example is the system of drift waves (DW, small scale) and zonal flows (ZF, large scale), which is important to our understanding of toroidal plasmas (Sagdeev *et al.*, 1978; Hasegawa *et al.*, 1979; Diamond *et al.*, 1998; Hinton and Rosenbluth, 1999; Smolyakov *et al.*, 1999; Champeaux and Diamond, 2001; Jenko *et al.*, 2001; Manfredi *et al.*, 2001; Li and Kishimoto, 2002; Diamond *et al.*, 2005b; Itoh *et al.*, 2006). Zonal flows are $\boldsymbol{E} \times \boldsymbol{B}$ drift flows on magnetic surfaces, with electric perturbation constant on the magnetic surface but changing rapidly across it. In the system of DW–ZF, the small-scale drift waves induce the transport of momentum (Reynolds stress). The divergence of the (off-diagonal) stress component amplifies the zonal flow shear. On the other hand, zonal shears stretch the drift wave packet. The coupling between them leads to the growth of zonal flows from drift wave turbulence. In this process, the energy of the drift waves is transferred to zonal flows, so that the small-scale fluctuation level and the associated transport is reduced. This is an important nonlinear process for the self-organization and confinement of toroidal magnetized plasmas, and will be described in detail in Volume 2 of this series of books. Other examples of disparate-scale interaction are also listed in Table 7.1.

In the following, we explain the disparate-scale nonlinear interaction by carefully considering the example of the system of the Langmuir wave and an acoustic wave. We examine two simplified limits. In Section 7.2, we consider the case where a (nearly) monochromatic Langmuir wave is subject to disparate-scale interaction. Through this example, the positive feedback loop between modulation of the envelope of small-scale oscillation and excitation of the acoustic wave is revealed. This loop crystallizes the elementary process that makes this particular coupling interesting and effective. The general methodology of envelope modulation is introduced, and illustrated using the particular example of the

Table 7.1. *Examples of disparate-scale interactions. Symbols* (s) *and* (l) *identify the small scale and large scale, respectively*

Example of system	Small scale	Large scale
(s) plasma wave and (l) acoustic wave	plasmon presssure (ponderomotive force)	refractions of plasmon ray by density perturbation
(s) drift wave (DW) and (l) zonal flow (ZF)	Reynolds stress on ZF	stretching and tilting of DW by shear
(s) MHD turbulence and (l) mean B field	mean induction of B (L) $\rightarrow$ dynamo	bending of mean field (l) by fluctuations (s)
(s) acoustic wave and (l) vortex	acoustic ponderomotive force	refraction of ray by vortex
(s) internal wave and (l) current	wave Reynolds stress	induced k-diffusions by random refraction of wave packets

Zakharov equations. By considering nonlinear evolution, the appearance of a singularity in a finite time (collapse) in the model is discussed (Section 7.4). The collapse of plasma wave packets is an alternate route to dissipation of the wave energy at a very small scale.

Study of the plain Langmuir wave is illustrative (for the understanding of self-focusing), but its applicability is limited. This is because, in reality, plasma waves are not necessarily (quasi-)plane waves, but more often appear in the form of packets or turbulence. Acoustic fluctuations are not a single coherent wave. Thus, theoretical methods to analyze the disparate-scale interaction in wave turbulence are necessary. In Section 7.3, the case of Langmuir wave turbulence is discussed. A theoretical approach, based on the quasi-particle picture, is explained. This method is a basic tool for studying plasma turbulence which exhibits disparate-scale interaction.

7.2 Langmuir waves and self-focusing

7.2.1 Zakharov equations

In this section, we illustrate the basic physics of the interaction between Langmuir waves (plasma waves, plasmons) and the ion sound wave (ion-acoustic wave) which induces self-focusing of the Langmuir waves. Plasma waves have wavelengths such that $k\lambda_{\mathrm{De}} < 1$. The Debye length λ_{De} is shorter than the characteristic wavelength of an ion sound wave. The scale lengths of these two kinds of waves are separated, but their nonlinear interaction constitutes one prototypical element of plasma dynamics. The basic physics that plays a role is:

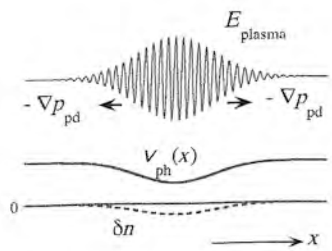

Fig. 7.3. Inhomogeneous plasma waves generate the pressure field for the mean electron dynamics. Density perturbation of the acoustic wave δn_e (dotted line) gives rise to the modulation of the refractive index of plasma waves (thick solid line).

Plasma wave $\rightarrow$ forms pressure field for acoustic waves.
Acoustic wave $\rightarrow$ density perturbation refracts plasma waves.

In the presence of plasma waves, electrons oscillate at the plasma wave frequency. The rapid electron motion associated with this wave produces the pressure field via ponderomotive force, discussed below. In other words, if the amplitude of the plasma wave is inhomogeneous, as is illustrated in Figure 7.3, then ambient electrons are repelled from the region of large amplitude. (The effective force, which arises from inhomogeneous and rapid oscillations, is known as 'ponderomotive force'.) Thus inhomogeneity of plasma waves causes pressure perturbations that couple to the acoustic wave. For the validity of this argument, the scale length of inhomogeneity of plasma waves must be longer than the wavelength of the plasma waves. The reciprocal influence of acoustic waves on plasma waves is caused by the fact that the dielectric function for plasma waves depends on the plasma density. Thus, the density perturbation associated with the ion sound wave causes modulation in the refractive index of plasma waves. The change of refractive index leads to modification of the plasma wave ray and intensity. These two processes constitute the loop that induces nonlinear instability of the plasma wave–acoustic wave system (Goldman, 1984; Zakharov, 1985; Robinson, 1997).

A heuristic description can be developed by application of the envelope formalism to plasma wave–acoustic wave interaction. Let us write the inhomogeneous plasma waves as,

$$\tilde{E} = E\left(\boldsymbol{x}, t\right) \boldsymbol{e}_0 \exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - \omega t\right), \tag{7.1}$$

where $E(\boldsymbol{x}, t)$ indicates the slowly-varying (in space and time) envelope, $\boldsymbol{e}_0$ denotes the polarization of the wave field, and $\exp(i\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$ is the rapidly oscillating plasma wave carrier. For plasma waves, the dispersion relation,

$$\omega^2 = \omega_{pe}^2 + \gamma_T k^2 v_{T,e}^2 \tag{7.2}$$

holds, where ω_{pe} is the plasma frequency,

$$\omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e},$$

γ_T is the specific heat ratio, and $v_{T,e}$ is the electron thermal velocity. In the dynamic expression, the dispersion relation (7.1) takes the form,

$$-\frac{\partial^2}{\partial t^2}\tilde{\boldsymbol{E}} = \omega_{pe}^2 \tilde{\boldsymbol{E}} - \gamma_T v_{T,e} \nabla^2 \tilde{\boldsymbol{E}}. \tag{7.3}$$

We now consider the two elements of the feedback loop in sequence.

(a) Influence of acoustic waves on plasma waves

When acoustic waves are present, the density perturbation is denoted by δn_e, so the plasma frequency becomes,

$$\omega_{pe}^2 = \omega_{p0}^2 (1 + \delta n) \quad \text{with} \quad \delta n \equiv \frac{\delta n_e}{n_{e,0}}. \tag{7.4}$$

Here, ω_{p0}^2 is defined at the unperturbed density $n_{e,0}$ as $\omega_{p0}^2 = 4\pi n_{e,0} e^2 / m_e$. Thus, in this case, the dynamical equation of plasma waves Eq.(7.3) takes the form,

$$-\frac{\partial^2}{\partial t^2}\tilde{\boldsymbol{E}} = \omega_{p0}^2 (1 + \delta n)\, \tilde{\boldsymbol{E}} - \gamma_T v_{T,e}^2 \nabla^2 \tilde{\boldsymbol{E}}. \tag{7.5}$$

If one substitutes Eq.(7.1) into Eq.(7.5), the rapidly varying terms balance, and the slowly varying envelope equation emerges as,

$$\frac{i}{\omega_{p0}} \frac{\partial}{\partial t} E + \lambda_{De}^2 \nabla^2 E = \delta n E, \tag{7.6}$$

where we use the relation $\lambda_{De}^2 = \gamma_T v_{T,e}^2 \omega_{p0}^{-2}$. In Eq.(7.6), the second term on the left-hand side is the diffraction term, which is caused by the dispersion of plasma waves. The modification of refraction, owing to the acoustic wave, appears on the right. Equation (7.6) describes how the envelope of the plasma waves is affected by the acoustic waves.

(b) Influence of plasma waves on acoustic waves

The influence of plasma waves on the acoustic wave is modelled by considering the contribution to the electron pressure from rapid oscillation by the plasma waves. In response to the plasma waves, electrons execute rapid oscillatory motion, but the kinetic energy of ions associated with this rapid oscillation is $m_e Z^2/m_i$-times smaller than that of electrons, owing to their heavier mass. (Here, Z is the charge per ion divided by the unit charge e.) In the slow-time scale, which is relevant to acoustic waves, the rapid electron oscillation by the plasma waves induces an effective wave or radiation pressure,

$$p_{\mathrm{pw}} \equiv \left\{ \frac{\partial}{\partial \omega} (\omega \epsilon) \bigg|_{\omega_{\mathrm{p0}}} \right\} \frac{|E|^2}{8\pi},$$

where ϵ is a dielectric function, which becomes unity in vacuum. The contribution from the response of ions to the rapid oscillation can be neglected. The energy density p_{pw} is that of the plasma waves, so electrons are repelled, on average (which is taken in a time scale longer than that of the plasma waves' frequency), from the region where p_{pw} takes a large value. This ponderomotive force (associated with the gradient of p_{pw}) induces ion motion on a slow timescale. In addition to the thermal pressure, p, p_{pw} also appears in the ion equation of motion, so,

$$m_i n_{i0} \frac{\partial}{\partial t} V = -\nabla \left(p + p_{\mathrm{pw}} \right).$$

The dynamic equation for the acoustic wave is then given as,

$$\left(\frac{\partial^2}{\partial t^2} - c_{\mathrm{s}}^2 \nabla^2 \right) \delta n = \frac{\nabla^2 |E|^2}{4\pi n_0 m_i}. \tag{7.7}$$

In deriving Eq.(7.7), the relation $\nabla^2 \left| \tilde{E} \right|^2 = \nabla^2 |E|^2$, and the approximation $\partial (\omega \epsilon) / \partial \omega |_{\omega_{\mathrm{p0}}} \simeq 1$ is applied to the plasma waves, and the relation $p_{\mathrm{T}} = p_0 (1 + \delta n)$ is used for (isothermal) acoustic waves. Equation (7.7) illustrates that the inhomogeneity of the envelope of plasma waves can excite the acoustic waves.

Equations (7.6) and (7.7) form a set of equations that describes the interaction of acoustic waves and the envelope of plasma waves. It is convenient to introduce dimensionless variables as,

$$\omega_{\mathrm{p0}} t \to t, \quad \lambda_{\mathrm{De}}^{-1} x \to x, \quad \frac{E}{\sqrt{4\pi n_0 T_e}} \to E. \tag{7.8}$$

(The new variable E is the oscillation velocity of an electron at the plasma wave frequency, normalized to the electron thermal velocity.) In these rescaled variables, the coupled equations (7.6) and (7.7) take the form,

$$\left(i\frac{\partial}{\partial t}+\nabla^2\right)E-\delta n E=0 \tag{7.9a}$$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{m_e}{m_i}\nabla^2\right)\delta n-\frac{m_e}{m_i}\nabla^2|E|^2=0. \tag{7.9b}$$

This set of equations is knows as the dimensionless 'Zakharov equations', and they are coupled envelope equations for the:

(a) plasma wave amplitude $E(\boldsymbol{x},t)$;
(b) density perturbation δn.

In the absence of nonlinear coupling, Eq.(7.9a) (i.e., with $\delta n \to 0$) becomes the Schrödinger equation for a free particle, and Eq.(7.9b) (with $|E| \to 0$) reduces to the acoustic wave equation. We again emphasize that in deriving the nonlinear coupling between classes of waves, the space-time scale separation is crucial.

7.2.2 Subsonic and supersonic limits

Depending on the velocity of envelope propagation, relative to the ion sound velocity, the nonlinearly coupled equations show different characters. Let us take the characteristic time scale τ and the characteristic scale length L for the envelope of plasma waves. The time derivative term in Eq.(7.9b) has the order of magnitude estimate $\partial^2/\partial t^2 \sim \tau^{-2}$, while the spatial derivative terms have $m_e m_i^{-1}\nabla^2 \sim m_e m_i^{-1}L^{-2}$. When evolution of the envelope is slow,

$$\tau^{-2} \ll m_e m_i^{-1}L^{-2} \text{ (i.e., rate of change} \ll c_s/L), \tag{7.10a}$$

the envelope modulation propagates much slower than the acoustic wave. This is the *subsonic* (*adiabatic*) limit. In contrast, if inequality

$$\tau^{-2} \gg m_e m_i^{-1}L^{-2} \text{ (i.e., rate of change} \gg c_s/L) \tag{7.10b}$$

holds, the modulation propagates much faster than the acoustic wave. This is the *supersonic* (*non-adiabatic*) limit.

Note that the plasma wave envelope propagates (in the linear response regime) at the group velocity, $v_g = \partial\omega_k/\partial k$. The dispersion relation $\omega_k = \omega_{\mathrm{pe}}\sqrt{1+k^2\lambda_{\mathrm{De}}^2}$ yields,

$$v_g = \frac{k\lambda_{\mathrm{De}}v_{\mathrm{T},e}}{\sqrt{1+k^2\lambda_{\mathrm{De}}^2}},$$

which is of the order of $k\lambda_{\mathrm{De}}v_{\mathrm{T},e}$ for the long wavelength limit $k\lambda_{\mathrm{De}} \ll 1$. Thus, the subsonic limit or supersonic limit depends on the wave number, in part. However, the propagation velocity also depends on the amplitude of plasma waves in a nonlinear regime.

7.2.3 Subsonic limit

In the subsonic limit, Eq.(7.10a), the inertia of ion motion (at slow time scale) is unimportant, and Eq.(7.7) is given by the balance between the ponderomotive force and the gradient in kinetic pressure. Thus, we have,

$$m_e m_i^{-1} \nabla^2 \left(\delta n + |E|^2 \right) = 0,$$

so

$$\delta n \cong - |E|^2 \tag{7.11}$$

is satisfied. Substituting Eq.(7.11) into Eq.(7.9a), the coupled Zakharov equations reduce to one combined equation,

$$\left(i\frac{\partial}{\partial t} + \nabla^2 \right) E + |E|^2 E = 0. \tag{7.12}$$

This is the so-called nonlinear Schrödinger equation (NLS equation), and is the adiabatic Zakharov equation. Note that density perturbations here are local depletions (i.e., $\delta n < 0$), and so are called cavitons.

7.2.4 Illustration of self-focusing

At this point, it is useful to recall the optical self-focusing problem, in order to understand the common physics content of focusing and the adiabatic Zakharov equation. Let us consider a light beam propagating in a nonlinear medium, in which the refractive index ($n^2 = c^2/v_{\mathrm{ph}}^2$) varies with the intensity of the light $|E|^2$, so,

$$n^2 = 1 + \Delta n \frac{|E|^2}{E_c^2},$$

where E_c^2 indicates the critical intensity above which the modification of the refractive index becomes apparent. The equation of light propagation is given as,

$$\nabla^2 \tilde{E} + \frac{\omega^2}{c^2} n^2 \tilde{E} = 0,$$

where ω is the frequency of the light and c is the speed of light in a vacuum. Substituting the wave and envelope modulation $\tilde{E} = E(\boldsymbol{x}, t)\, \boldsymbol{e}_0 \exp(i k \cdot z - \omega t)$ into this wave propagation equation, we have,

$$\left(2ik\frac{\partial}{\partial z} + \nabla^2\right) E + \Delta n k^2 |E|^2 E = 0, \qquad (7.13)$$

where z is taken in the direction of the propagation and $k^2 = \omega^2 c^{-2}$. Terms of $O\left(k^{-2}\, \partial^2 E/\partial z \partial r\right)$ have been neglected. This is the NLS equation, describing the self-focusing of light in a nonlinear medium. The physics of self-focusing is the change of the phase speed owing to the intensity of the light,

$$v_{\rm ph}^2 = \frac{c^2}{n^2} = \frac{c^2}{1 + \Delta n |E|^2}.$$

Thus, when $\Delta n > 0$ holds, the phase velocity becomes *lower* in regions of *high intensity*. This is illustrated by considering propagation of phase fronts (i.e., contoured surface of constant phase) in a beam of finite width. (Rays are perpendicular to the phase front.) As is illustrated in Figure 7.4, the phase front (iso-phase surface) is deformed in the region of high intensity. When $\Delta n > 0$ holds, the phase front lags behind in the region of high intensity, so that the phase front becomes concave. The local propagation direction of the wave is perpendicular to the phase front. When the phase front becomes concave, the local propagation directions are no longer parallel, but instead tend to focus. As a result, the light beam focuses itself. The peak intensity of light becomes higher and higher as the light propagates. It suggests formation of a singularity in the light intensity field, unless an

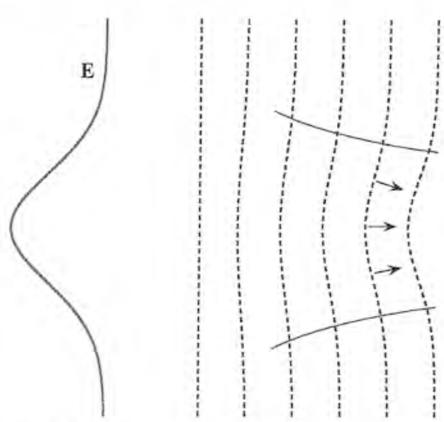

Fig. 7.4. Propagation of a light beam in a nonlinear medium with $\Delta n > 0$. Phase fronts are shown by dotted lines, and a local direction of propagation is denoted by arrows. Thin lines illustrate knees of the beam intensity profile.

additional mechanism alters this self-focusing process. The problem of singularity formation will be discussed later in this chapter.

7.2.5 *Linear theory of self-focusing*

The dynamics of self-focusing in the NLS equation can be analyzed using a linear analysis. The NLS equation (7.12) is a complex equation, so that the envelope functions in terms of two fields, i.e., the amplitude and the phase,

$$E = A \exp(i\varphi). \tag{7.14}$$

Both the amplitude A and the phase φ are (slow) functions of space and time. Substituting the complex form of E into Eq.(7.12), the real and imaginary parts reduce to a set of coupled equations for the (real) fields A, φ,

$$\frac{1}{A}\frac{\partial}{\partial t}A + \nabla^2\varphi + \frac{2\nabla\varphi\cdot\nabla A}{A} = 0, \tag{7.15a}$$

$$\frac{\partial}{\partial t}\varphi + (\nabla\varphi)^2 - \left(\frac{\nabla^2 A}{A} + |A|^2\right) = 0. \tag{7.15b}$$

The second term on the left-hand side of Eq.(7.15a) (the $\nabla^2\varphi$ term) shows that the amplitude increases if the phase front is concave ($\nabla^2\varphi < 0$) as is illustrated in Figure 7.4. In Eq.(7.15b), the fourth term on the left-hand side (the $|A|^2$ term) indicates that the larger amplitude causes variation in the phase. Thus the spatial variations in the intensity field induce bending of the phase front in Figure 7.4. These features induce self-focusing of the plasma waves.

A systematic analysis of the growth of amplitude modulation for plasma waves can be performed by linearizing Eqs.(7.15). We put,

$$A = A_0 + \tilde{A}, \quad \varphi = \tilde{\varphi}$$

in Eq.(7.15), and retain the linear terms in $\tilde{A}$, $\tilde{\varphi}$,

$$\frac{\partial}{\partial t}\tilde{A} + A_0\nabla^2\varphi = 0 \tag{7.16a}$$

$$A_0\frac{\partial}{\partial t}\tilde{\varphi} - \nabla^2\tilde{A} - A_0^2\tilde{A} = 0. \tag{7.16b}$$

Putting the perturbation in the form,

$$\left(\tilde{A}, \tilde{\varphi}\right) \propto \exp\left(i\boldsymbol{q}\cdot\boldsymbol{x} - i\Omega t\right),$$

where q and Ω denote slow spatio-temporal variation of envelope, we find the dispersion relation,

$$\Omega^2 = -q^2\left(A_0^2 - q^2\right). \tag{7.17}$$

That is, the perturbation grows if $A_0^2 > q^2$. The term $q^2 A_0^2$ on the right of Eq.(7.17) is the destabilizing term for the self-focusing, and the q^4 term denotes the effect of diffraction, which spreads or blurs modulation. Thus, the NLS equation describes the amplification of the peak plasma wave intensity $|E|^2$ and the perturbed density of long wavelength, if the amplitude of plasma waves exceeds the threshold, $A_0 > q$, i.e., in a dimensional form,

$$\frac{|E|^2}{4\pi n_0 T_e} > q^2\lambda_{\mathrm{De}}^2. \tag{7.18}$$

(The condition is interpreted that the electron oscillation velocity induced by the plasma wave is larger than $q\lambda_{\mathrm{De}}v_{\mathrm{T},e}$. It is more easily satisfied if q is smaller.) This criterion shows a competition between the self-attraction of plasma waves (through depleting background plasma density on a large scale) and the spreading of plasma waves associated with their diffraction.

Equation (7.18) indicates that the plane plasma waves are, at any amplitude, unstable against amplitude modulation by choosing the long wavelength of modulation. In reality, there is a lower bound of the light intensity for instability. For instance, system size limits the lowest allowable value for q; in addition, small but finite dissipation (which is ignored for transparency of the argument) can suppress the instability, because the maximum growth rate in Eq.(7.17), $A_0^2/2$, scales as $|E|^2$.

When linear theory predicts the instability to occur, the central problem is how the focusing evolves. Such nonlinear evolution is discussed at the end of Section 7.4, which deals with Langmuir collapse.

7.3 Langmuir wave turbulence

The Zakharov equations in Section 7.2 illustrate the essence of self-focusing of plasma waves by disparate scale interactions in plasmas. This set of equations was derived for a plane plasma wave interacting with narrow-band ion acoustic waves. In more general circumstances, plasma waves are excited as turbulence, and the acoustic spectrum might also be composed of broad band wavelengths. The spatio-temporal evolution of the envelope of Langmuir turbulence may or may not be slower than the acoustic speed.

In order to analyze the evolution of plasma wave turbulence by disparate scale interactions, it is illuminating to use the quasi-particle description. The scale separation between the carrier plasma waves and the modulator (ion sound waves) is exploited. As is explained in Chapter 5, the Manley–Rowe relation holds for three-wave interactions. When a scale separation in frequency and wave number applies for the plasma waves and acoustic waves, the action density of plasma waves is invariant during the interaction with an acoustic wave. Of course, the action density of plasma waves changes in time on account of nonlinear interaction among plasma waves, which occurs on the range of their common scales. These effects can be incorporated into theory as a collision operator on the right-hand side of the quasi-particle kinetic equation.

7.3.1 Action density

The energy density and action density (in the phase space) of plasma waves are introduced as,

$$E_k \equiv \frac{\partial}{\partial \omega}(\omega \epsilon)\bigg|_{\omega_k} \frac{\left|\tilde{\boldsymbol{E}}_k\right|^2}{8\pi} \quad \text{and} \quad N = \frac{E_k}{\omega_k}, \tag{7.19}$$

where $\tilde{\boldsymbol{E}}_k$ is the electric field of the plasma wave at the wave number k, and the dispersion relation of plasma waves is given by $\omega_k^2 = \omega_{pe}^2 + \gamma_T k^2 v_{T,e}^2$. Now, $N(\boldsymbol{k}, \boldsymbol{x}, t)$ is a population density of waves (excitons). Regarding the variables $(\boldsymbol{k}, \boldsymbol{x}, t)$ of $N(\boldsymbol{k}, \boldsymbol{x}, t)$, $\boldsymbol{k}$ stands for the wave vector of the carrier wave (that is the short scale, corresponding to the plasma waves), and $\boldsymbol{x}$ and t denote a slow scale which appears in the envelope.

7.3.2 Disparate scale interaction between Langmuir turbulence and acoustic turbulence

The coupling between Langmuir turbulence and acoustic turbulence is studied employing the quasi-particle approach. In derivation of the model, we take the limit where the time and spatial scales of these two kinds of turbulence are well separated. As is illustrated in Eq.(7.9), an expansion parameter that leads to scale separation is, in this case, essentially the ratio of electron mass to ion, $\sqrt{m_e/m_i} \ll 1$. The perturbations of the Langmuir turbulence field (i.e., radiation pressure, etc.) are expressed in terms of the action density $N = \omega_k^{-1} E_k$ defined in Eq.(7.19), where $\tilde{\boldsymbol{E}}_k$ has slow spatio-temporal variations. The acoustic waves are characterized by the density and velocity perturbations $\tilde{n}$ and $\tilde{\boldsymbol{V}}$. The quantities $\tilde{n}$ and $\tilde{\boldsymbol{V}}$ have spatio-temporal dependencies which are slow compared to $\boldsymbol{k}$ and

ω_k of Langmuir wave turbulence. Under the circumstance of scale separation, the quantity $N(\boldsymbol{k}, \boldsymbol{x}, t)$, the number distribution of waves, is conserved along the trajectory. That is, the number of quanta of the $(\boldsymbol{k}, \omega_k)$ component of the Langmuir wave moves in the $(\boldsymbol{k}, \boldsymbol{x})$-space as a '*particle*' interacting with the field of acoustic waves. The conservation relation,

$$\frac{\mathrm{d}N(\boldsymbol{k}, \boldsymbol{x}, t)}{\mathrm{d}t} = 0,$$

in the presence of the acoustic waves is rewritten as,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\partial N}{\partial t} + \left(\boldsymbol{v}_{\mathrm{g}} + \tilde{\boldsymbol{V}}\right) \cdot \frac{\partial N}{\partial \boldsymbol{x}} - \frac{\partial}{\partial \boldsymbol{x}}\left(\omega_k + \boldsymbol{k} \cdot \tilde{\boldsymbol{V}}\right) \cdot \frac{\partial N}{\partial \boldsymbol{k}} = 0. \tag{7.20}$$

Note that, in addition to interaction with ion acoustic waves, the excitation (by an instability or external supply), damping (by such as the Landau damping for $\omega \simeq k v_{\mathrm{T},e}$) and energy transfer among Langmuir waves through self-nonlinear interactions can take place. These processes, which occur on the scale of Langmuir waves, lead to the rapid, non-adiabatic evolution of Langmuir wave action density. The rate of change in $N(\boldsymbol{k}, \boldsymbol{x}, \boldsymbol{t})$ through these mechanisms is schematically written $\Gamma(k, \omega; x, t)$. (As for the wave field, the slow spatio-temporal dependence is expressed in terms of $\boldsymbol{x}, t$.) Thus, the evolution equation of Langmuir wave action under the influence of the acoustic waves is written as,

$$\frac{\partial N}{\partial t} + \left(\boldsymbol{v}_{\mathrm{g}} + \tilde{\boldsymbol{V}}\right) \cdot \frac{\partial N}{\partial \boldsymbol{x}} - \frac{\partial}{\partial \boldsymbol{x}}\left(\omega_k + \boldsymbol{k} \cdot \tilde{\boldsymbol{V}}\right) \cdot \frac{\partial N}{\partial \boldsymbol{k}} = \Gamma(k, \omega; x, t; N)\, N. \tag{7.21}$$

This equation states that the wave quanta N in the phase space $N(\boldsymbol{x}, \boldsymbol{k})$, having a finite life time $-\Gamma^{-1}$, follow trajectories determined by the eikonal equation,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial \omega_k}{\partial \boldsymbol{k}} + \tilde{\boldsymbol{V}} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\frac{\partial}{\partial \boldsymbol{x}}\left(\omega_k + \boldsymbol{k} \cdot \tilde{\boldsymbol{V}}\right).$$

The dynamical equation (7.21) is simplified in order to study the response against the acoustic waves. In the presence of long-scale perturbations, the wave frequency is modified such that,

$$\omega_k = \omega_{k0} + \tilde{\omega}_k,$$

(ω_{k0} is given in the absence of acoustic waves), and the *unperturbed orbit of quasi-particles* may be defined as,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial \omega_{k0}}{\partial_k} = \boldsymbol{v}_{\mathrm{g}}, \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\frac{\partial}{\partial \boldsymbol{x}}\omega_{k0}. \tag{7.22}$$

The term $\Gamma(k, \omega; x, t; N)\, N$ includes linear terms of N (e.g., due to the linear damping/growth or collisional damping, etc.) and nonlinear terms of N

(e.g., owing to the nonlinear interactions in small scales). Therefore, as in the Chapman–Enskog approach, the mean distribution $\langle N \rangle$ is determined by the relation,

$$\boldsymbol{v}_{\mathrm{g}} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{x}} - \frac{\partial \omega_k}{\partial \boldsymbol{x}} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}} = \Gamma\left(k, \omega; x, t; \langle N \rangle\right) \langle N \rangle .$$

The deviation from the mean, $\tilde{N}$, is induced by coupling with the acoustic wave. The distribution N and the term $\Gamma\left(k, \omega; x, t; N\right) N$ are rewritten as,

$$N = \langle N \rangle + \tilde{N}, \tag{7.23a}$$

$$\Gamma\left(k, \omega; x, t; N\right) N = \Gamma\left(k, \omega; x, t; \langle N \rangle\right) \langle N \rangle - \hat{\Gamma} \tilde{N} + \cdots, \tag{7.23b}$$

where $\hat{\Gamma} \tilde{N}$ is the first-order correction (so that $\hat{\Gamma}$ is independent of $\tilde{N}$). In the usual circumstances, where the linear growth (growth rate: γ_{L}) is balanced by the quadratic nonlinearity between different plasma waves (self-nonlinearity), the estimate,

$$\hat{\Gamma} \simeq \gamma_{\mathrm{L}}$$

holds. Keeping the first-order terms with respect to $\tilde{N}$, Eq.(7.21) becomes,

$$\frac{\partial \tilde{N}}{\partial t} + \boldsymbol{v}_{\mathrm{g}} \cdot \frac{\partial \tilde{N}}{\partial \boldsymbol{x}} - \frac{\partial}{\partial \boldsymbol{x}} \omega_{k0} \cdot \frac{\partial \tilde{N}}{\partial \boldsymbol{k}} + \hat{\Gamma} \tilde{N}$$
$$= -\left(\frac{\partial \tilde{\omega}_k}{\partial \boldsymbol{k}} + \tilde{\boldsymbol{V}}\right) \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{x}} + \frac{\partial}{\partial \boldsymbol{x}} \left(\tilde{\omega}_k + \boldsymbol{k} \cdot \tilde{\boldsymbol{V}}\right) \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}}. \tag{7.24}$$

The analogy between this equation and the Boltzmann equation is evident in light of the Chapman–Enskog expansion. The plasma waves, the amplitude of which is modulated by acoustic waves, is illustrated in Figure 7.5(a). In this circumstance,

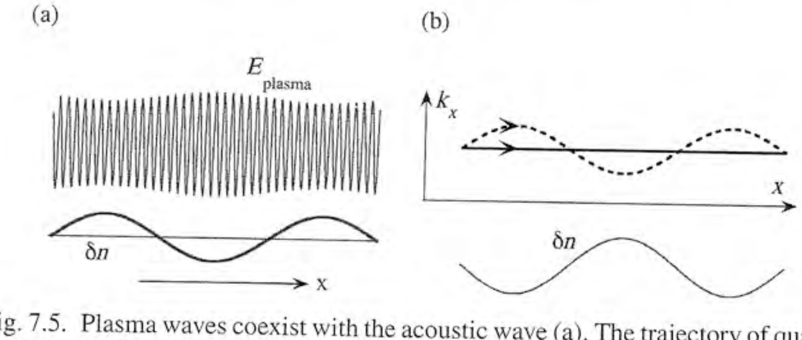

Fig. 7.5. Plasma waves coexist with the acoustic wave (a). The trajectory of quasi-particles of plasma waves in the presence of the acoustic wave (dashed line) and the unperturbed trajectory (solid line), (b). (Schematic drawing.)

the trajectory of the quasi-particle (plasma waves) is shown by the dashed line in Figure 7.5(b), and the unperturbed orbit (7.22) is illustrated by the solid line. Equation (7.20) indicates that the action is unchanged by acoustic waves along the perturbed orbit. As was the case for solving the Boltzmann and Vlasov equations, the contribution from the acoustic waves is separated in the right-hand side of Eq.(7.24). Treating this as a source, the action of plasma wave quasi-particles is calculated by integrating along the unperturbed trajectory of quasi-particles. The reaction of the Langmuir wave turbulence on the acoustic wave takes place by the pressure perturbation on electrons i.e. $-\nabla \int \mathrm{d}k\, \omega_k N$. Adding this term to the pressure perturbation associated with the sound waves, one has,

$$\frac{\partial^2}{\partial t^2}\tilde{n} - \nabla^2 \tilde{n} = -\nabla^2 \frac{2}{m_i} \int \mathrm{d}k\, \omega_k N. \tag{7.25}$$

The set of equations (7.24) and (7.25) describes the evolution of the coupled Langmuir–acoustic turbulence through disparate scale interactions. From the viewpoint of investigating mean scales (the scale of acoustic waves), the evolution equation for the Langmuir waves (7.21) is effectively an energy equation for *subgrid scales*. In this analogy, the effect of plasma waves (*unresolved* on the scale of Eq.(7.25)) on *resolved* acoustic waves occurs through the radiation stress term on the right-hand side of Eq.(7.25).

7.3.3 Evolution of the Langmuir wave action density

In order to study the mutual interaction between plasma waves and acoustic waves, we consider that the modulation of density in the acoustic wave fluctuations, $\delta n = \tilde{n}/n_0$, is the first-order term, and put the action density of plasmons, Eq.(7.23a), $N = \langle N \rangle + \tilde{N}$, where $\tilde{N}$ is of the order of $\delta n = \tilde{n}/n_0$.

Consideration of energy

The average action density $\langle N \rangle$ evolves much slower than the acoustic wave fluctuations. In the absence of acoustic waves, the stationary solution, which is determined by the term $\Gamma\,(k, \omega; x, t; N)$, gives $\langle N \rangle$ as is explained in (7.23b). By coupling with acoustic waves, N evolves slowly in time. In interacting with acoustic waves, the action N is conserved. Thus the change of energy density of plasma waves E_k follows,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_k = N \frac{\mathrm{d}}{\mathrm{d}t} \omega_k.$$

Noting the relations,

$$\frac{\mathrm{d}\omega_k}{\mathrm{d}t} = \frac{\partial \omega_k}{\partial \boldsymbol{k}} \cdot \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}, \qquad \frac{\partial \omega_k}{\partial \boldsymbol{k}} = \boldsymbol{v}_{\mathrm{g}}$$

and the dynamics of the refractive index by density perturbation $\mathrm{d}\boldsymbol{k}/\mathrm{d}t = -\partial/\partial\boldsymbol{x}\left(\omega_{\mathrm{p}0}\delta n\right)$, one has,

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k = -N\omega_{\mathrm{p}0}\boldsymbol{v}_{\mathrm{gr}}\frac{\partial}{\partial\boldsymbol{x}}\delta n.$$

Putting $N = \langle N \rangle + \tilde{N}$ into this relation, and noting that the correlation $\langle N \rangle \delta n$ vanishes in a long-time average but that of $\tilde{N}\delta n$ may survive, we have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle E_k \rangle = -\omega_{\mathrm{p}0}\boldsymbol{v}_{\mathrm{gr}} \cdot \left\langle \tilde{N}\frac{\partial}{\partial\boldsymbol{x}}\delta n \right\rangle. \tag{7.26}$$

This relation illustrates that the change of energy of plasma waves, which is transferred to acoustic waves, is given by the correlation $\left\langle \tilde{N}\, \partial\delta n/\partial\boldsymbol{x} \right\rangle$.

Wave kinetic equation of action density

For this purpose, we analyze the case that the plasma wave turbulence is homogeneous (in the unperturbed state), and the Doppler shift by the ion fluid motion $\boldsymbol{k} \cdot \tilde{\boldsymbol{V}}$ is smaller than the effect of the modulation of the refractive index, $\tilde{\omega}_k$. The relation,

$$\frac{\partial\tilde{\omega}_k}{\partial\boldsymbol{x}} = \nabla\tilde{n}\left(\frac{\partial\tilde{\omega}_k}{\partial n_e}\right)$$

is employed. Under this circumstance, putting Eq.(7.26) into Eq.(7.21), together with Eq.(7.24), yields the responses of $\tilde{N}$ and $\langle N \rangle$ to the acoustic waves as,

$$\frac{\partial\tilde{N}}{\partial t} + \boldsymbol{v}_{\mathrm{gr}} \cdot \frac{\partial\tilde{N}}{\partial\boldsymbol{x}} - \frac{\partial}{\partial\boldsymbol{x}}\omega_{k0} \cdot \frac{\partial\tilde{N}}{\partial\boldsymbol{k}} + \hat{\Gamma}\tilde{N} = \frac{\partial\omega_k}{\partial n_e}\nabla\tilde{n} \cdot \frac{\partial\langle N \rangle}{\partial\boldsymbol{k}}, \tag{7.27a}$$

$$\frac{\partial\langle N \rangle}{\partial t} + \langle \Gamma N \rangle = \frac{\partial}{\partial\boldsymbol{k}} \cdot \left\langle \frac{\partial\omega_k}{\partial n_e}\nabla\tilde{n}\tilde{N} \right\rangle. \tag{7.27b}$$

A similar way of thinking leads the back interaction of plasma quasi-particles on the acoustic wave, Eq.(7.25) into the form,

$$\frac{\partial^2}{\partial t^2}\tilde{n} - \nabla^2\tilde{n} = -\nabla^2\frac{2}{m_i}\int \mathrm{d}k\,\omega_{k0}\tilde{N}. \tag{7.28}$$

The coupled system of Eqs.(7.27) and (7.28) has a clear similarity to the Vlasov–Maxwell equations which describe the evolutions of particle distribution function and fields. The correspondence between the disparate scale interaction and the Vlasov–Maxwell system is summarized in Table 7.2.

Table 7.2. *Analogy and correspondence between the Vlasov–Maxwell system and the disparate scale interaction in Langmuir turbulence are summarized*

	Vlasov plasma	Quasi-particles and disparate scale interaction
Particle	electron, ion	plasma wave (plasmon)
Velocity	particle velocity	group velocity of plasma wave, or $\boldsymbol{k}$
Distribution	$f(\boldsymbol{x}, \boldsymbol{v})$	$N(\boldsymbol{x}, \boldsymbol{v}_g)$, or $N(\boldsymbol{x}, \boldsymbol{k})$
Field	electro(magnetic) field	ion acoustic wave
Change *of velocity*	acceleration by fields	modification of $\boldsymbol{v}_g$ or $\boldsymbol{k}$ by refraction
Dynamical equation for particles	Vlasov equation	wave kinetic equation for N
Equation for field	Poisson equation	equation for acoustic wave

7.3.4 Response of distribution of quasi-particles

It is convenient to use the normalized density perturbation δn, so we take the spatio temporal structure of acoustic wave fluctuations to be,

$$\delta n = \frac{\tilde{n}}{n} = \sum_{q,\Omega} \delta n_{q,\Omega} \exp\left(i\boldsymbol{q}\cdot\boldsymbol{x} - i\Omega t\right), \quad \tilde{N} = \sum_{q,\Omega} \tilde{N}_{q,\Omega} \exp\left(i\boldsymbol{q}\cdot\boldsymbol{x} - i\Omega t\right). \tag{7.29}$$

As is the case in subsection 7.2.5, $\boldsymbol{q}$ and Ω stand for the slow spatio-temporal variation associated with acoustic waves. For transparency of argument, we take,

$$\frac{\partial \omega_k}{\partial n_e} = \frac{1}{2}\frac{\omega_{p0}}{n_0},$$

where n_0 and ω_{p0} are the density and plasma frequency at unperturbed state. Equations (7.27) and (7.29) immediately give the response,

$$\tilde{N}_{q,\Omega} = -\frac{\delta\tilde{n}\omega_{p0}}{\Omega - \boldsymbol{q}\cdot\boldsymbol{v}_g + i\hat{\Gamma}}\boldsymbol{q}\cdot\frac{\partial\langle N\rangle}{\partial\boldsymbol{k}}. \tag{7.30}$$

Applying quasi-linear theory to calculating the mean evolution, as explained in Chapter 3, then yields the change of plasmon energy as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle E_k\rangle = -\omega_{p0}^2 \sum_{q,\Omega} \boldsymbol{v}_{gr}\cdot\boldsymbol{q}\frac{i\left|\delta n_{q,\Omega}\right|^2}{\Omega - \boldsymbol{q}\cdot\boldsymbol{v}_{gr} + i\hat{\Gamma}}\boldsymbol{q}\cdot\frac{\partial\langle N\rangle}{\partial\boldsymbol{k}}. \tag{7.31}$$

When the self-interaction process of plasma waves is weaker than the decorrelation due to the dispersion of waves, (i.e., $\tau_{ac} < \tau_c, \tau_T$) one may take,

$$\frac{i}{\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{g} + i\hat{\Gamma}} \simeq \pi\delta\left(\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{g}\right)$$

where $\delta\left(\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{g}\right)$ is Dirac's delta function. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle E_k \right\rangle = -\omega_{p0} \sum_{q,\Omega} \pi\delta\left(\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{g}\right) \left|\delta n_{q,\Omega}\right|^2 \boldsymbol{v}_{g} \cdot \boldsymbol{q}\boldsymbol{q} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}}. \tag{7.32}$$

This describes the relation between the population density $\langle N \rangle$ and the direction of energy transfer between plasma waves and acoustic waves. Since the group velocity $\boldsymbol{v}_{g} = \partial\omega_k/\partial\boldsymbol{k} = \gamma_T \omega_k^{-1} v_{T,e}^2 \boldsymbol{k}$ is positive for plasma waves, the energy evolution rate satisfies the condition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle E_k \right\rangle < 0 \quad \text{if} \quad \frac{\partial \langle N \rangle}{\partial k} > 0. \tag{7.33}$$

The energy is transformed from plasma wave quasi-particles to acoustic waves in the case of a population inversion $\partial \langle N \rangle / \partial k > 0$. That is, the acoustic waves grow in time at the expense of plasma wave quasi-particles. This resembles the inverse cascade, in that the long-wavelength modes accumulate energy from short-wavelength perturbations. It is important to stress that the difference is that the energy transfer takes place through the disparate scale interactions, not by local couplings. The energy from plasma waves is *directly* transferred to acoustic waves, without exciting intermediate scale fluctuations.

The transfer of energy from plasma waves to acoustic waves occurs through diffusion of the plasma wave quasi-particles. Substituting Eq.(7.30) into Eq.(7.27b), one obtains the evolution of the mean action density as,

$$\frac{\partial \langle N \rangle}{\partial t} + \langle \Gamma N \rangle = \frac{\partial}{\partial \boldsymbol{k}} \cdot \sum_{q,\Omega} \boldsymbol{q} \frac{i\omega_{p0}^2 \left(\delta\tilde{n}\right)^2}{\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{g} + i\hat{\Gamma}} \boldsymbol{q} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}}. \tag{7.34}$$

This is a diffusion equation for $\langle N \rangle$, i.e.,

$$\frac{\partial \langle N \rangle}{\partial t} + \langle \Gamma N \rangle = \frac{\partial}{\partial \boldsymbol{k}} \cdot \boldsymbol{D} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}} \tag{7.35a}$$

with,

$$\boldsymbol{D} = \sum_{q,\Omega} \frac{i\omega_{p0}^2 \left(\delta\tilde{n}\right)^2}{\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{g} + i\hat{\Gamma}} \boldsymbol{q}\boldsymbol{q}. \tag{7.35b}$$

It is evident that the total action density $\int \mathrm{d}k \, \langle N \rangle$ is conserved in the interaction with acoustic waves, because operating with $\int \mathrm{d}k$ to the right-hand side of Eq.(7.35a) vanishes. The total action $\int \mathrm{d}k \, \langle N \rangle$ is determined by the balance between the source and sink, i.e., $\int \mathrm{d}k \, \langle \Gamma N \rangle$. This redistribution of action density leads to the energy exchange between plasma waves and acoustic waves. Operating with $\int \mathrm{d}k \, \omega_k$ on Eq.(7.35a), the energy transfer rate, $(\partial/\partial t) \int \mathrm{d}k \, \omega_k \langle N \rangle$, induced by acoustic waves is seen to be equal to,

$$-\int \mathrm{d}k \, \frac{\partial \omega_k}{\partial \boldsymbol{k}} \cdot \boldsymbol{D} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}},$$

which is equivalent to Eq.(7.32). Thus, energy relaxation occurs if $\boldsymbol{v}_{\mathrm{g}} \cdot \boldsymbol{D} \cdot \partial \langle N \rangle / \partial \boldsymbol{k} > 0$ holds.

The diffusion of $\langle N \rangle$ and the energy transfer are explained in Figure 7.6, using an example of a one-dimensional problem. (The vector $\boldsymbol{q}$ is in one dimension.) The density $\langle N \rangle$ is subject to flattening and turns into the one shown by dashed lines. The areas 'A' and 'B' in this figure are equal. However, the frequency is higher in the domain 'A' than that in 'B', so that the energy content in 'A' is larger than that in 'B'. The difference between energies in 'A' and 'B' is converted into acoustic waves.

The $\boldsymbol{k}$-space diffusion coefficient for plasma wave quasi-particles is evaluated as,

$$D \simeq \left\langle q^2 \omega_{\mathrm{p}}^2 \delta n^2 \right\rangle \tau_{\mathrm{ac}}, \quad \tau_{\mathrm{ac}} = \left| \Delta \left(\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{\mathrm{g}} + i \hat{\Gamma} \right) \right|^{-1}, \qquad (7.36)$$

where $\left| \Delta \left(\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_{\mathrm{g}} + i \hat{\Gamma} \right) \right|$ is the width of the resonance of quasi-particles with the acoustic wave field. Equations (7.32) or (7.35) indicate that relaxation

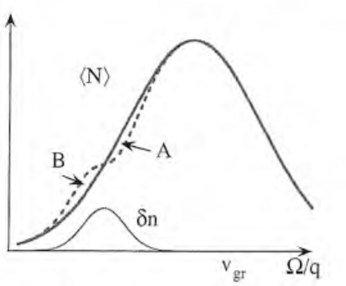

Fig. 7.6. Relaxation of the quasi-particle action density owing to interaction with acoustic waves (in a one-dimensional problem). The density $\langle N \rangle$ is subject to flattening and turns into the one shown by a dashed line. The areas 'A' and 'B' in this figure are equal. However, the frequency of plasma waves is higher in domain 'A' than that in 'B', so that the energy content in 'A' is larger than that in 'B'. The difference between energies in 'A' and 'B' is converted into acoustic waves.

of the plasma wave action occurs if the velocity of a quasi-particle coincides with the phase velocity of acoustic waves, $\Omega \simeq \boldsymbol{q} \cdot \boldsymbol{v}_g$. The group velocity of plasma waves $\boldsymbol{v}_g = \gamma_T \omega_k^{-1} v_{T,e}^2 \boldsymbol{k}$ and the dispersion of the acoustic wave $\Omega = c_s q/\sqrt{1 + q^2\lambda_{De}^2} \simeq c_s q$ lead to the resonance condition,

$$\frac{\boldsymbol{q} \cdot \boldsymbol{k}}{qk} \simeq \frac{c_s}{\gamma_T v_{T,e}} \frac{1}{k\lambda_{De}}. \tag{7.37}$$

In a one-dimensional situation $\boldsymbol{q} \cdot \boldsymbol{k} \simeq qk$, this condition is satisfied for the wavelength $k\lambda_{De} \simeq O\left(\sqrt{m_e/m_i}\right)$. Oblique propagation of acoustic wave $\boldsymbol{q} \cdot \boldsymbol{k} \ll qk$ allows substantial resonance for the cases of $k\lambda_{De} > \sqrt{m_e/m_i}$. The subsonic limit, Eq.(7.10a), holds for the case $\Omega \ll \boldsymbol{q} \cdot \boldsymbol{v}_g$, i.e., $k\lambda_{De} \ll \sqrt{m_e/m_i}$, or very oblique propagation of modulation. (In the configuration of oblique propagation of modulation by the acoustic wave, the growth rate of modulation is smaller.)

Evolution of the mean action density is given by the induced diffusion equation. Resonant diffusion with the ion acoustic wave is irreversible. The origin of the irreversibility is discussed here. In the limit of small self-nonlinear interaction, $\Gamma \to 0$, the evolution equation of quasi-particles, Eq.(7.20), preserves time-reversal symmetry, as in the Vlasov equation. As is explained in Chapter 3, the irreversibility of resonant quasi-linear diffusion is ultimately rooted in orbit chaos. In the case of quasi-particle interaction with acoustic wave fields, *ray chaos* occurs by the overlap of multiple resonances at $\Omega \simeq \boldsymbol{q} \cdot \boldsymbol{v}_g$ for various values of the velocity of quasi-particles $\boldsymbol{v}_g$. The dispersion of the acoustic wave is weak, $\Omega = c_s q/\sqrt{1 + q^2\lambda_{De}^2}$, and $\Omega \simeq q c_s$ for a wide range of Ω. The overlap of wave–quasi-particle resonance can happen as follows. Although the dispersion of acoustic waves is weak, there is a small but finite dispersion for acoustic waves. In addition, acoustic waves may be subject to a damping process owing to the kinetic interaction of ions. This provides a finite bandwidth for acoustic wave fluctuations, and thus induces the overlap of acoustic wave–quasi-particle resonances. The other route to overlapping is the oblique propagation of multiple acoustic waves, by which the resonance condition $\Omega \simeq \boldsymbol{q} \cdot \boldsymbol{v}_g$ can be satisfied for quasi-particles with a wide range of group velocity magnitude and direction.

7.3.5 Growth rate of modulation of plasma waves

The growth rate of the envelope of self-focusing plasma waves is derived for the subsonic limit in the preceding subsections. More general cases can be investigated by use of the quasi-particle approach. Substituting the response of quasi-particles (7.30) into Eq.(7.28), the dispersion relation of the acoustic wave field is given as,

$$-\Omega^2 + q^2 c_s^2 = \frac{q^2}{m_i}\omega_{p0}^2 \int \mathrm{d}k \, \frac{1}{\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_g + i\hat{\Gamma}} \boldsymbol{q} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}}. \tag{7.38}$$

In the perturbation analysis where the right-hand side of Eq.(7.38) is smaller than $q^2 c_s^2$, one has,

$$\Omega = q c_s + i \frac{\pi q^2}{2m_i} \frac{\omega_{p0}^2}{q c_s} \int \mathrm{d}k \, \delta\left(\Omega - \boldsymbol{q} \cdot \boldsymbol{v}_g\right) \boldsymbol{q} \cdot \frac{\partial \langle N \rangle}{\partial \boldsymbol{k}}, \tag{7.39}$$

(for the case of a small self-nonlinearity term $\hat{\Gamma} \to 0$). Instability is possible if,

$$\frac{\partial \langle N \rangle}{\partial \boldsymbol{k}} > 0 \quad \text{at} \quad \Omega \simeq \boldsymbol{q} \cdot \boldsymbol{v}_g.$$

7.3.6 Trapping of quasi-particles

In order to proceed to understand nonlinear evolution of the modulation of plasma wave turbulence, the perturbation of the quasi-particle trajectory is illustrated here. Let us consider the motion of a quasi-particle in a single and coherent acoustic wave perturbation. The case where the lifetime of the quasi-particle (through a self-nonlinear mechanism) is small, $\Gamma \to 0$, is explained. This situation corresponds to the 'collisionless limit', i.e., the Vlasov plasma.

When the quasi-particle is subject to a longer-wavelength perturbation (ion sound wave in this case), the particle trajectory is deformed and trapping of the orbit in the eikonal phase space takes place. For transparency of argument, we consider again the simple limit in Section 7.3.3 (which lead to Eq.(7.26)). That is, we consider the case where the plasma wave turbulence is homogeneous (in the unperturbed state), and the Doppler shift by the ion fluid motion $\boldsymbol{k} \cdot \tilde{\boldsymbol{V}}$ is smaller than the effect of the modulation of the refractive index, $\tilde{\omega}_k$. The dynamic equations of quasi-particles in the presence of large-scale fields, $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \partial\omega_k/\partial\boldsymbol{k} + \tilde{\boldsymbol{V}}$ and $\mathrm{d}\boldsymbol{k}/\mathrm{d}t = -\partial\left(\tilde{\omega}_k + \boldsymbol{k} \cdot \tilde{\boldsymbol{V}}\right)/\partial\boldsymbol{x}$, are simplified to $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \partial\omega_k/\partial\boldsymbol{k}$ and $\mathrm{d}\boldsymbol{k}/\mathrm{d}t = -\partial\tilde{\omega}_k/\partial\boldsymbol{x}$. In the presence of the long-scale perturbations, $n = n_0 + \tilde{n}$, the relation,

$$\partial\tilde{\omega}_k/\partial\boldsymbol{x} = \nabla\tilde{n}\left(\partial\omega_k/\partial n_e\right)$$

is employed. Noting the relation $\partial\omega_k/\partial n_e = (1/2)n_0^{-1}\omega_{p0}$ for the plasma wave, one has,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_g \simeq \gamma_T \omega_{p0}^{-1} v_{T,e}^2 \left(k_0 + \tilde{k}\right) \tag{7.40a}$$

$$\frac{\mathrm{d}\tilde{k}}{\mathrm{d}t} = -\frac{\nabla\tilde{n}}{n_0}\omega_{p0}, \tag{7.40b}$$

where k_0 denotes the wave number that satisfies the resonance condition $v_g = \Omega/q$ and $\tilde{k}$ is the modulation of the wave number by the acoustic mode. It is evident from this relation that the quasi-particles undergo a bounce motion (in the frame which is moving together with the large-scale wave). If one explicitly puts,

$$\frac{\tilde{n}}{n} = -\delta n \cos\left(qx - \Omega t\right),$$

(i.e., the origin is taken at the trough of the density perturbation of the acoustic wave), the relative displacement of the quasi-particle in the frame moving with the acoustic mode, $\xi = x - \Omega t/q$, obeys the equation,

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -\delta n \gamma_T v_{T,e}^2 q \sin\left(q\xi\right). \tag{7.41}$$

That is, the orbit Eq.(7.40) is given by the elliptic function. Figure 7.7 illustrates the trajectories of quasi-particles. Trapped orbits and transiting orbits are separated

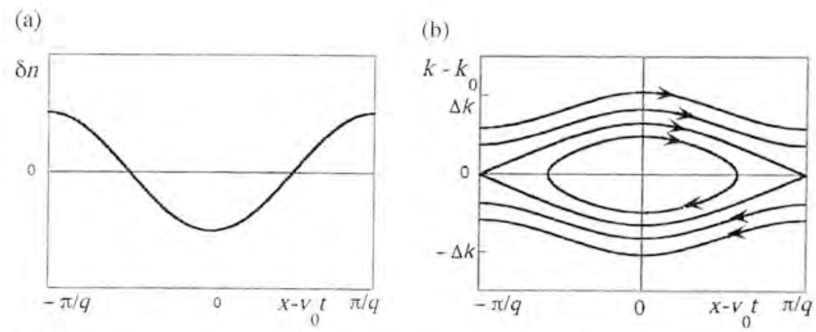

Fig. 7.7. Trapping of a quasi-particle in the trough of a long-wavelength perturbation. Here $v_0 = \Omega/q$ indicates the phase velocity of the long-wavelength mode and k_0 denotes the wave number that satisfies the resonance condition $v_g = \Omega/q$. (a) shows the density perturbation of the acoustic mode, and (b) illustrates the trajectory of the quasi-particle (plasmon) in the phase space.

by a separatrix. Near the trough of the large-scale density perturbation, the bounce frequency is given as,

$$\omega_\mathrm{b} = \sqrt{\gamma_T \delta n}\, v_{\mathrm{T},e} q. \tag{7.42}$$

(Bounce frequency is the inverse of rotation time of a trapped quasi-particle.) The bounce frequency is in proportion to the square-root of the density perturbation of the acoustic waves. Multiplying $\mathrm{d}\xi/\mathrm{d}t$ to the right and left-hand sides of Eq.(7.41), and integrating once in time, one has an integral of motion,

$$\frac{1}{2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)^2 = \delta n \gamma_T v_{\mathrm{T},e}^2 \cos(q\xi) + C, \tag{7.43}$$

where C is an integration constant. The trajectory forms a separatrix for the quasi-particle orbit with $C - \delta n \gamma_T v_{\mathrm{T},e}^2$. The largest modulation of the velocity of the quasi-particles is given as,

$$\Delta v_\mathrm{g} = 2\sqrt{\delta n \gamma_T}\, v_{\mathrm{T},e}, \tag{7.44}$$

or, in terms of the wave number,

$$\Delta k = 2\sqrt{\frac{\delta n}{\gamma_T}}\frac{\omega_{\mathrm{p}0}}{v_{\mathrm{T},e}}.$$

7.3.7 Saturation of modulational instability

The modulational instability of plasma wave turbulence is saturated when the perturbation amplitude becomes high. The analogy between the systems of:

(i) quasi-particle (plasma wave) and perturbed field (sound wave);
(ii) particle and perturbed electric field in Vlasov plasma

immediately tells that the modulational instability can be saturated through:

(a) flattening of the distribution function through phase space diffusion;
(b) nonlinear Landau damping by coupling to stable sound waves;
(c) trapping of quasi-particles in the trough of the field.

To identify the dominant process, the key parameters are the bounce frequency of the quasi-particle in the trough of the large-scale field, ω_b, and the dispersion in the wave number of the quasi-particle due to the large-scale fields, Δk. A typical example of Δk is the width of island in the phase space for the quasi-particle orbit.

When multiple acoustic waves exist so that resonance can occur at various values of velocity Ω/q, the separation of two neighbouring phase velocities $\Delta(\Omega/q)$

is introduced. If the separation between two phase velocities $\Delta(\Omega/q)$ is larger than the variation of the velocity of the quasi-particle in a trough of a large-scale field, $\left(\partial v_g/\partial k\right)\Delta k$, i.e.,

$$\Delta(\Omega/q) > \left(\partial v_g/\partial k\right)\Delta k, \tag{7.45a}$$

the quasi-particle orbit is trapped (if the lifetime is long enough) in a trough of one wave. In contrast, if the separation is smaller than the change in quasi-particle velocity,

$$\Delta(\Omega/q) < \left(\partial v_g/\partial k\right)\Delta k, \tag{7.45b}$$

so that 'island-overlapping' of resonances occurs. We introduce a 'Chirikov parameter' for a quasi-particle dynamics in long-wavelength perturbation fields as,

$$\mathcal{S} = \frac{\left(\partial v_g/\partial k\right)\Delta k}{\Delta(\Omega/q)}. \tag{7.46}$$

Thus, the island overlapping condition is given by $\mathcal{S} > 1$.

The other key parameter is the Kubo number of quasi-particles. Here, Kubo number $\mathcal{K}$ is the ratio between the lifetime of the quasi-particle (which is limited by the self-nonlinear effects of short-wavelength perturbations) and the bounce time of the quasi-particles in the trough of longer wavelength perturbations,

$$\mathcal{K} = \frac{\omega_b}{\Gamma}. \tag{7.47}$$

When $\mathcal{K}$ is much smaller than unity, a quasi-particle loses its memory before completing the circumnavigation in a trough of large-scale waves. Thus trapping does not occur. In contrast, if $\mathcal{K}$ is much larger than unity, the trapping of quasi-particles plays the dominant role in determining the nonlinear evolution. Note that $\mathcal{K}$ is closely related to the Strouhal number, familiar from fluid turbulence. The Strouhal number $\mathcal{S}_t$ is given by,

$$\mathcal{S}_t = \frac{\tilde{V}\tau_c}{l_c}.$$

If we take $\tilde{V}/l_c \sim 1/\tau_b$, i.e., to identify the 'bounce time' with an 'eddy circulation time', $\mathcal{S}_t$ may be re-written,

$$\mathcal{S}_t = \frac{\tau_c}{\tau_b},$$

which is essentially the Kubo number.

In the $(\mathcal{K}, \mathcal{S})$ diagram, important nonlinear processes are summarized (Diamond *et al.*, 2005b; Balescu, 2005). First, when the modulation amplitude is small and only one wave is considered $(\mathcal{K}, \mathcal{S} \to 0)$, the method of modulational parametric instability applies, and the instability criterion of Section 7.3.5 is deduced.

When $\mathcal{K}$ is small but $\mathcal{S}$ is larger than unity, ray chaos occurs. That is, the motion of the quasi-particle becomes stochastic (without being trapped in one particular trough), and diffuse in phase space. The quasi-linear diffusion approach (Chapter 3) applies. The evolution equation for the action density is given as,

$$\frac{\partial}{\partial t} \langle N \rangle - \frac{\partial}{\partial k} D_k \frac{\partial \langle N \rangle}{\partial k} = -\Gamma \langle N \rangle , \tag{7.48}$$

where D_k is the diffusion coefficient of the quasi-particle in the field of acoustic wave turbulence (Eq.(7.36)).

In the other extreme limit, i.e., $\mathcal{K} \to \infty$ but $\mathcal{S}$ is small, a quasi-particle moves (without decorrelation) along its perturbed orbit. Thus, ultimately a BGK state (Bernstein *et al.*, 1957; Kaw *et al.*, 1975, 2002) is approached. The integral of motion, C in Eq.(7.43), is given as,

$$C(x_0, k_0) = \frac{1}{2}\left(\gamma_T \frac{v_{\mathrm{T},e}^2}{\omega_{\mathrm{p}0}} k_0 - \frac{\Omega}{q}\right)^2 - \delta n \gamma_T v_{\mathrm{T},e}^2 \cos(q x_0),$$

where (x_0, k_0) is the position of the quasi-particle in the phase space at $t = 0$. Each trajectory in Figure 7.7(b) is characterized by the constant of motion $C(x_0, k_0)$. The distribution of the action density $N(x, k)$ is constant along this trajectory in a stationary state. By use of these integrals of motion, an exact solution for the distribution function is given in the form,

$$N(x, k_x) = N\left(C\left(x_0, k_{x0}\right)\right). \tag{7.49}$$

When $\mathcal{S}$ is small, the beat wave excitation by modulated quasi-particles couples to the (more stable) acoustic waves. Then, nonlinear Landau damping of quasi-particles is the means for saturation of the perturbations. Coherent structure may be sustained by the balance between the (linear) modulational instability and nonlinear Landau damping of quasi-particles.

In the intermediate regime of $\mathcal{K}, \mathcal{S} \sim 1$, turbulent trapping of quasi-particles occurs. The various theoretical approaches are summarized in Figure 7.8.

7.4 Collapse of Langmuir turbulence

7.4.1 Problem definition

The nonlinear mechanism through disparate scale interaction is shown to generate large-scale perturbations. These might be observed as "mesoscale structures"

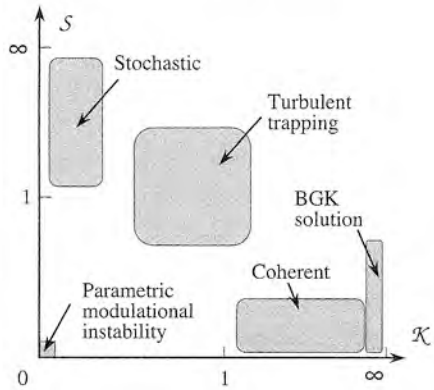

Fig. 7.8. Parameter domains for various theoretical approaches.

when global quantities are observed. This mechanism is, at the same time, the origin of a new route to enhanced energy dissipation at small scales. The standard route of energy dissipation occurs through the cascade into finer-scale perturbations, for which the understanding from Kolmogorov's analysis is very powerful. The evolution of self-focusing (which is induced by disparate scale interactions) can generate the compression of plasma wave energy into a small area. This phenomenon is known as the 'collapse' of plasma waves (Armstrong *et al.*, 1962; Bespalov and Talanov, 1966).

Recall, the problem of singularity in fluid turbulence. The dissipation rate per unit volume ϵ satisfies the relation $\epsilon = \left\langle \nu \left(\nabla V\right)^2 \right\rangle$, where ν is the (molecular) viscosity. If the energy is continuously injected into the system and is dissipated at microscale, it is plausible (but unproven) that the dissipation rate ϵ remains independent of ν, stays in the vicinity of the mean value (possibly with fluctuations in time) and does not vanish. Thus, the relation $(\nabla V)^2 = \epsilon/\nu$ implies that $|\nabla V| \to \infty$ in the invicid limit $\nu \to 0$. Thus, singularity formation is expected to occur. This hypothesis of singularity formation in Navier–Stokes turbulence has not yet been proven vigorously, but lies at the heart of turbulence physics. ("Singularity" does not, of course, indicate divergence of the velocity, but rather the formation of steep velocity gradients. In reality, the model of continuum for the fluid breaks down; new dynamics for energy dissipation occurs at the scale where the fluid description no longer holds.)

The collapse of the plasma wave is explained here, in order to illuminate an alternative (in comparison with the Kolmogorov cascade process) self-similar route to dissipation.

7.4.2 Adiabatic Zakharov equation

In order to illuminate the physics of collapse through disparate-scale interaction, we employ the adiabatic Zakharov equation, Eq.(7.12),

$$\left(i\frac{\partial}{\partial t}+\nabla^2\right)E+|E|^2E=0.$$

(The envelope is denoted by E, and the normalization is explained in Eq.(7.8).) The Zakharov equation indicates that coherent plasma waves can be subject self-focusing if the criterion Eq.(7.18) is satisfied. Once the focusing starts, the local enhancement of wave intensity will further accelerate the formation of localized structures, so that a singularity may form.

The nonlinear state of the focusing depends critically on the dimensionality of the system. In fact, the NLS equation in one-dimensional space is known to be integrable: the nonlinear stationary solutions (soliton solutions) were found to be,

$$E(x,t)=E_0\mathrm{sech}\left(\sqrt{\frac{1}{2}}E_0x\right)\exp\left(\frac{i}{2}E_0^2t\right). \tag{7.50}$$

(Noting the Galilean invariance of the adiabatic Zakharov equation (7.12), one may say that solutions with finite propagation velocity are easily constructed.)

In a three-dimensional system, focusing continues and collapse occurs. This process is explained in this subsection, but a qualitative explanation is given before going into details. Let us consider the situation where the wave field of high intensity $|E|^2$ is localized in a region with a size l. The conservation of the total plasma wave energy, which is in proportion to $|E|^2\,l^d$, requires that $|E|^2\,l^d$ remain constant (here, d is the number of space dimensions). When the size of the bunch changes, $l\to l'$, the squared intensity changes as $|E|^2\to\left(l/l'\right)^d$. The diffraction term increases as $\left(l/l'\right)^2$. Thus the nonlinear attraction term works more strongly than the diffraction term in the three-dimensional case $d=3$, so that contraction of the system size is not restricted. In the one-dimensional case, the increment of diffraction is larger upon contraction, so that the self-focusing stops at finite amplitude. In that case, diffraction balances self-focusing, thus producing a soliton structure.

7.4.3 Collapse of plasma waves with spherical symmetry

The envelope with spherical symmetry, $E(r,t)$, in the adiabatic Zakharov equation satisfies,

$$i\frac{\partial}{\partial t}E+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}E+|E|^2E=0. \tag{7.51}$$

With the help of an analogy with quantum physics, we can identify integrals of this equation. First is the conservation of the total plasmon number. If one introduces the 'flux',

$$F = i\left(E\,\partial E^*/\partial r - E^*\,\partial E/\partial r\right),$$

Eq.(7.51) immediately gives the conservation relation for intensity distribution,

$$\partial\,|E|^2/\partial t + r^{-2}\,\partial\left(r^2F\right)/\partial r = 0.$$

Thus one finds that the total intensity of the wave (i.e., the total number of plasmons, I_1) is conserved,

$$I_1 = \int_0^\infty \mathrm{d}r\, r^2\,|E|^2\,. \tag{7.52a}$$

The total Hamiltonian I_2 is also conserved in time, i.e.,

$$I_2 = \int_0^\infty \mathrm{d}r\, r^2\left(\left|\frac{\partial E}{\partial r}\right|^2 - \frac{1}{2}\,|E|^4\right) \tag{7.52b}$$

is a constant of motion. According to the analogy of quantum physics, the first term and the second term in the integral are the kinetic energy and (attractive) potential energy, respectively. The second term on the right-hand side of Eq.(7.52b) is the 'attractive potential', which increases when the amplitude of the wave becomes larger. Thus, the 'total energy' I_2 (for which $|\partial E/\partial r|^2$ is the 'kinetic energy density' and $|E|^4$ is the 'potential energy') is positive for a small amplitude wave, and I_2 is negative for large amplitude waves. In other words, the small amplitude wave is considered to be a 'positive energy state' and the large amplitude wave is considered to be a 'negative energy state'.

By use of these two constants of motion, the evolution of the mean radius,

$$\left\langle r^2\right\rangle = (I_1)^{-1}\int_0^\infty \mathrm{d}r\, r^4\,|E|^2\,,$$

can be studied. Noting the conserved form for the intensity,

$$\partial\,|E|^2/\partial t + r^{-2}\,\partial\left(r^2F\right)/\partial r = 0,$$

one has the relation,

$$\partial^2\left(r^2\,|E|^2\right)/\partial t^2 + r^{-2}\,\partial\left(r^4\,\partial F/\partial t\right)/\partial r - 2r\,\partial F/\partial t = 0.$$

Therefore, the mean squared radius obeys the evolution equation,

$$I_1 \frac{\partial^2}{\partial t^2}\left\langle r^2 \right\rangle = 2\int_0^\infty \mathrm{d}r\, r^3 \frac{\partial}{\partial t} F.$$

The integral of the right-hand side of this relation is calculated by use of the NLS equation. Thus, we have the identity,

$$\frac{\partial^2}{\partial t^2}\left\langle r^2 \right\rangle = 8\frac{I_2}{I_1} - \frac{2}{I_1}\int_0^\infty \mathrm{d}r\, r^2 \left|E\right|^4, \tag{7.53a}$$

that is,

$$\frac{\partial^2}{\partial t^2}\left\langle r^2 \right\rangle < 8\frac{I_2}{I_1}. \tag{7.53b}$$

The mean squared radius $\left\langle r^2 \right\rangle$ must satisfy the condition,

$$\left\langle r^2 \right\rangle < 8\frac{I_2}{I_1}t^2 + \frac{\partial}{\partial t}\left\langle r^2 \right\rangle\Big|_0 t + \left\langle r^2 \right\rangle\Big|_0. \tag{7.54}$$

The integral I_1 is positive definite. Therefore, if the initial value of I_2 is negative, the mean squared radius approaches zero after a finite time. The mean-squared radius $\left\langle r^2 \right\rangle$ is positive definite, by definition, so that the solution must encounter a singularity in a finite time. The time for the collapse, τ_{collapse}, where $\left\langle r^2 \right\rangle = 0$ holds, can be calculated from Eq.(7.54). For instance, starting from a stationary initial condition, $\frac{\partial}{\partial t}\left\langle r^2 \right\rangle\big|_0 = 0$, one has the bound,

$$\tau_{\mathrm{collapse}} < \sqrt{\frac{-I_1}{8I_2}\left\langle r^2 \right\rangle\Big|_0}. \tag{7.55}$$

In the large amplitude limit, where the second term is much larger than the first term in the integrand of Eq.(7.52b), the ratio $-I_1/I_2$ is of the order of $|E|^{-2}$. The normalization is shown in Eq.(7.8), i.e., $\omega_{\mathrm{p0}}t \rightarrow t$, $\lambda_{\mathrm{De}}^{-1}x \rightarrow x$, and E is the oscillation velocity of an electron at the plasma wave frequency ($\tilde{v}_{\mathrm{pw}}$), normalized to the electron thermal velocity. Then, the estimate $\tau_{\mathrm{collapse}} \sim \sqrt{|E|^{-2}\left\langle r^2 \right\rangle\big|_0}$ (apart from a factor of order unity) can be interpreted as the time for the onset of collapse as,

$$\tau_{\mathrm{collapse}} \sim \frac{l}{\tilde{v}_{\mathrm{pw}}}, \tag{7.56}$$

where l is the initial size of the hump of plasma waves. (An example from numerical calculation is given in Figure 7.9.)

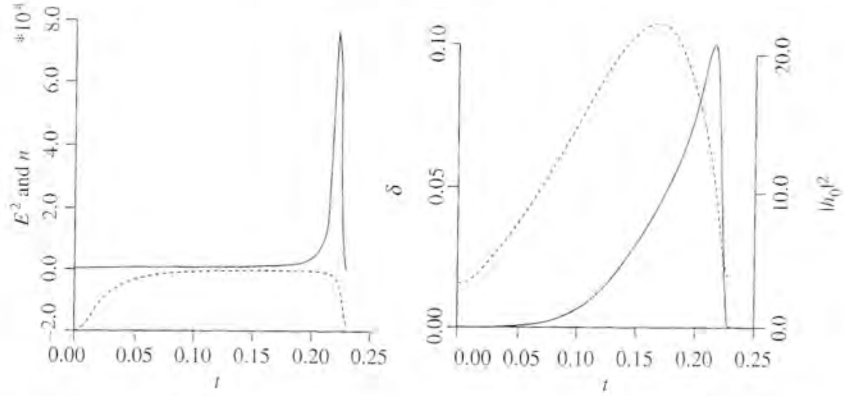

Fig. 7.9. Example of spherical collapse of Langmuir waves. Intensity of waves $|E|^2$ and perturbation density (dashed line) at the centre $r = 0$ are given in (a). (Parameters are: pump wave intensity $|E_0| = 5$, $m_i/m_e = 2000$. Normalization units for time, length, electric field and perturbed number density are $(3m_i/2m_e)\,\omega_{pe}^{-1}$, $(3/2)\sqrt{m_i/m_e}\lambda_{De}$, $8\sqrt{\pi n_0 T_e m_e/3m_i}$ and $n_0 4m_e/3m_i$, respectively.) In the early phase of the collapse, the eigenmode with lowest eigenvalue $\mathbf{e}_0(\mathbf{x}, t)$, the amplitude of which is $h_0(t)$, grows dominantly. Amplitude $|h_0(t)|^2$ and radius of collapsing state $\delta(t)$ are shown in (b); $\delta(t)$ is defined by the radius that maximizes $r^2|\mathbf{e}_0(\mathbf{r}, t)|^2$. [quoted from (Dubois *et al.*, 1988)].

Note that the relation between $\partial^2\langle r^2\rangle/\partial t^2$ and I_2/I_1, like Eq.(7.52), is derived for the two-dimensional ($d = 2$) and the one-dimensional ($d = 1$) cases. One obtains after some manipulation,

$$\frac{\partial^2}{\partial t^2}\left\langle r^2\right\rangle = 8\frac{I_2}{I_1}, \quad (d = 2) \tag{7.57a}$$

$$\frac{\partial^2}{\partial t^2}\left\langle r^2\right\rangle = 8\frac{I_2}{I_1} + \frac{2}{I_1}\int_0^\infty \mathrm{d}r\ |E|^4, \quad (d = 1) \tag{7.57b}$$

(where the weight of volume element r^2 in the integrand of Eq.(7.52) is appropriately adjusted). We see that the proportionality between $\partial^2\langle r^2\rangle/\partial t^2$ and I_2/I_1 holds for the 2D case. Collapse occurs in this case, if the *initial total energy* is negative $I_2 < 0$. In contrast, for the one-dimensional problem, $d = 1$, collapse is prohibited by the second term on the right-hand side of Eq.(7.57b), even if the condition $I_2 < 0$ is satisfied. For an arbitrary negative value of I_2, the second term on the right-hand side eventually overcomes the first term if focusing continues. Thus, contraction must stop at a finite size. Comparison between these three cases is illustrated schematically in Figure 7.10. (Note that the physics of singularity formation works in other turbulence, too. For instance, extension to singularity formation

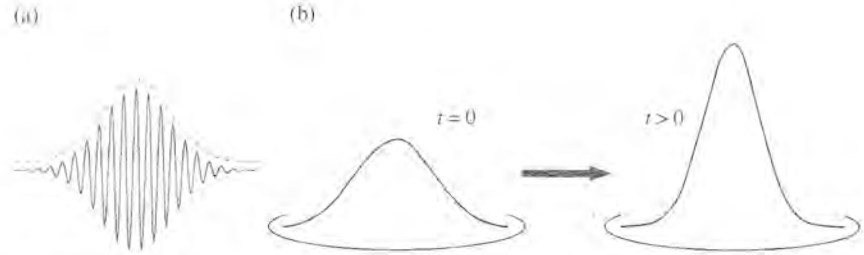

Fig. 7.10. Schematic description of the cases of 1D (a), 2D and 3D (b).

in electron-temperature-gradient-driven turbulence was discussed in (Gurcan and Diamond, 2004).)

7.4.4 *Note on 'cascade versus collapse'*

The collapse via disparate scale nonlinear interaction is a new route to the dissipation of wave energy. This process is compared to the Kolmogorov cascade process. Equation (7.56) shows that the time of collapse is shorter if the intensity of the Langmuir wave is higher, and is longer if the size of an original spot is larger. Such a dimensional argument is also available for the Kolmogorov cascade.

We revisit the relation (7.56) for the case of cascade. The scale length of the largest eddy is evaluated by the energy input scale $l_{\mathrm{E}} = k_{\mathrm{E}}^{-1}$, where k_{E} is defined by the condition that most of energy resides near k_{E}, i.e.,

$$K \cong K_0 \epsilon^{2/3} \int_{k_{\mathrm{E}}}^{\infty} l^{-5/3} \mathrm{d}k,$$

where ϵ is the dissipation rate of the energy per unit volume and K is the kinetic energy density. From this consideration, we have $k_{\mathrm{E}} = \epsilon K^{-3/2}$, that is,

$$l_{\mathrm{E}} = \epsilon^{-1} K^{3/2}.$$

The characteristic time for energy dissipation is evaluated by considering the sequence of cascades which make an eddy into smaller eddies. Let us consider a process that the n-th eddy (size l_n, velocity v_n) is broken into the $(n+1)$-th eddy (Figure 2.12(b)),

$$l_{n+1} = \alpha l_n,$$

(where l_n is given by l_{E} at $n = 0$). The time for the cascade from the n-th eddy to the $(n+1)$-th eddy, induced by the nonlinearity $\boldsymbol{v} \cdot \nabla$, is given by $\tau_n = l_n / v_n$, thus,

$$\textit{time for the cascade to the n-th eddy} = \sum_{n=0}^{n} \tau_n = \sum_{n=0}^{n} \frac{l_n}{v_n}.$$

By use of Richardson's law, $v_n = \epsilon^{1/3} l_n^{1/3}$, one has

$$\sum_{n=0}^{n} \tau_n = \epsilon^{-1/3} \sum_{n=0}^{n} l_n^{2/3} = \epsilon^{-1/3} l_0^{2/3} \sum_{n=0}^{n} \alpha^{2n/3}.$$

Taking the limit of $n \to \infty$, one has,

$$\tau_{\text{cascade}} = \sum_{n=0}^{\infty} \tau_n = \frac{l_{\text{E}}^{2/3}}{\epsilon^{-1/3}} \frac{1}{1-\alpha^{2/3}} \sim \frac{1}{1-\alpha^{2/3}} \frac{l_{\text{E}}}{\tilde{v}}, \tag{7.58a}$$

where $\tilde{v} = \epsilon^{-1/3} l_{\text{E}}^{-1/3}$ is the turbulent velocity. The time for the energy cascade is approximately given by the eddy turn-over time, τ_{eddy},

$$\tau_{\text{eddy}} = \frac{l_{\text{E}}}{\sqrt{K}} = \frac{l_{\text{E}}}{\tilde{v}}. \tag{7.58b}$$

Comparing Eqs.(7.56) and (7.58), one observes a similarity which appears in the dependence as,

$$\tau \sim l/\tilde{v}.$$

The difference is, however, noticeable: the time given by (7.57) scales (up to the microscopic scale where dissipation works). In contrast, the equation (7.56) indicates that the energy is concentrated in the area of focus, which has a small radius. The cascade is considered as a stationary state, while the alternative path such as collapse is an explosive and dynamical phenomenon, which resembles steepening en route to shock formation.